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Algebraic and Geometrical Methods in Topology

Conference on Topological Methods in Algebraic Topology SUNY Binghamton, October 3-7, 1973

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INTRODUCTION

This volume consists of the proceedings of the supplementary program of the Five Day Regional Conference on Topological Methods in Algebraic Topology - A History of Classifying Spaces held at the State University of New York at Binghamton, October 3 - 7, 1973. It was the first conference held on this subject in the world.

The conference was supported by a grant from the National Science Foundation. The principle speaker of the conference was Professor E. E. Floyd, Robert C. Taylor Professor of Mathematics, The University of Virginia. His series of ten lectures should appear as a publication of the Conference Board of the Mathematical Sciences (U.S.A.). The supplementary program was an extremely important part of the conference and is represented by the manuscripts herein. The State University of New York provided a grant from its program "Conversations in the Disciplines" which partially supported the supplementary program. We are indebted not only to the National Science Foundation and the State University of New York but also to all who participated in the conference and contributed so much to its success.

The supplementary program covered a wide variety of topics which assisted in making the conference an extremely interesting one. All lectures in this program were given by invitation. These lectures were on topics of current research interest in algebraic, geometric, and differential topology. They have varying degrees of relationship to the central theme. Some attempt has been made to group them by subject as indicated in the table of contents.

Papers in Section I involve various aspects of homotopy theory with the paper of Stasheff directly related to the conference theme.

Section II consists of two papers in category theory as related to algebraic topology. The work represented in Section III concerns a variety of topics all in the area of manifold and differential topology. The papers in Section IV and V concern aspects of geometric topology with infinite dimensional manifolds represented in Section IV and differential geometry represented in Section V.

We deeply regret that it is impossible to reprint the following papers which were presented at our conference and which represented an important part of the supplementary program. They are as follows:

Gluck, Krigelman, and Singer; "The Converse to the Gauss-Bonnet Theorem in PL".

Singer, David; "Preassigning curvature on the Two-Sphere".

These will appear in the Journal of Differential Geometry.

Cohen, Marshall; "A Proof that Simple-Homotopy Equivalent Polyhedra are Stably Homeomorphic".

This paper will appear in the Michigan Mathematical Journal.

Heller, Alex; "Adjoint Functors and Bar Constructions".

This paper will appear in Advances in Mathematics.

We are most grateful to Jeanne Osborne for her assistance in the careful preparation for the conference and for the thorough manner in which she handled administrative details. We are particularly indebted to Althea Benjamin for the superb typing of the manuscripts.

We would like to acknowledge the invaluable editorial assistance rendered by Ross Geoghegan and Patricia McAuley of the Department of Mathematical Sciences, State University of New York at Binghamton, who read many of the manuscripts and provided various other editorial services.

We are no less appreciative of the assistance of Naomi Bar-Yosef, Barbara Lamberg, and Elizabeth Newton.

Finally, we are indebted to Springer-Verlag for publishing these proceedings and, in particular, to Alice Peters for her supervisory role.

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State University of New York at Binghamton

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ALGEBRAIC TOPOLOGY - A HISTORY OF CLASSIFYING SPACES

October 3 - 7, 1973

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The following were among the faculty and graduate students of the State University of New York at Binghamton who were participants in the Conference:

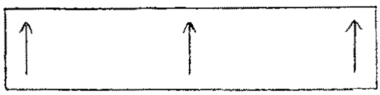
David Edwards
Ross Geoghegan
Louie Mahony
Patricia McAuley
Prabir Roy
Alan Coppola
Steve Dibner
Ron Fintushel
Eric Robinson

PARALLEL TRANSPORT AND CLASSIFICATION OF FIBRATIONS

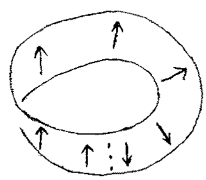
by

James D. Stasheff[1]

The simplest example of parallel transport is the field of (parallel) vertical vectors on $I \times I$:



and the simplest non-trivial example occurs when we form this strip into a Moebius band:

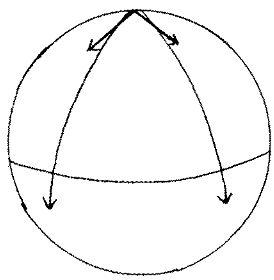


clearly distinguishing the Moebius band from the cylinder.

The idea of parallel transport originates in differential geometry where geometric structure such as curvature is revealed by parallel

[1]Research supported in part by the NSF.

transporting tangent vectors along curves:



Essentially the same idea occurs in covering space theory where a loop in the space covered determines a deck transformation or permutation of the sheets of the covering. [Veblen and Whitehead] suggested the greater generality of fibre bundles as a setting. We shall look at fiber spaces as well.

We begin formally.

Provisional Definition: For a fibre space $F \longrightarrow E \longrightarrow X$, a (parallel) transport is a map

$$\tau : F \times \Omega X \longrightarrow F$$

such that

1) the trivial loop acts as the identity

2) each loop acts as a homotopy equivalence

3) τ is transitive (i.e., $\tau(f, \lambda + \mu) = \tau(\tau(f,\lambda),\mu)$) or reasonably close to it.

Classically and intuitively we would expect strict transitivity: transporting the fibre around one loop and then another should be the same as transporting it around the sum of the two loops. For fibre spaces we lack such precision as we can see by constructing τ from the Covering Homotopy Property.

Consider

$$\begin{array}{ccc} F \times \Omega X & \xrightarrow{f_o} & E \\ \downarrow & & \downarrow \\ F \times \Omega X & \xrightarrow[g_t]{} & X \end{array}$$

where $f_o(y,\lambda) = y$ and $g_t(y,\lambda) = \lambda(t)$. The CHP gives us

$f_t: F \times \Omega X \longrightarrow E$ with $f_1: F \times \Omega X \longrightarrow F$; in fact, we can assume $f_t(y,e) = y$ where e is the trivial loop. We set $\tau = f_1$ and achieve 1 and 2.

The lifting f_t is not unique, but any two are homotopic. (They are homotopic within E to f_o by a homotopy whose image in X is homotopy trivial and thus the homotopy can be deformed to be fibre preserving, i.e., f_1 and f_1' are homotopic in F.) The same reason applies to show $\tau(\tau \times 1) \simeq \tau \quad (1 \times m): F \times \Omega X \times \Omega X \longrightarrow F$ where m is loop addition [Hilton].

One can in fact say more, but we need a language with which to say it. One approach is to consider the adjoint map $\text{ad}\,\tau : \Omega X \longrightarrow F^F$. (We will not worry about the function space topology but rather always use continuity in reference to τ rather than $\text{ad}\,\tau$). The transitivity of τ is equivalent to the multiplicativity of $\text{ad}\,\tau$. The homotopy condition above is equivalent to $\text{ad}\,\tau$ being an H-map. In general for maps of one associative H-space to another we have the notion of strong homotopy multiplicativity.

<u>Definition</u>. If Y and Z are topological monoids, a map $f:Y \longrightarrow Z$ is s.h.m. (strongly homotopy multiplicative) if any of the following conditions are satisfied:

a) There exist maps $f_n : Y^n \times I^{n-1} \longrightarrow Z$ such that $f_1 = f$ and

$$f_n(y_1,\cdots,y_n,t_1,\cdots,t_{n-1}) =$$

$$f_{n-1}(\cdots,y_i y_{i+1},\cdots,\hat{t}_i,\cdots) \quad \text{if} \quad t_i = 0$$

$$f_i(y_1,\cdots,y_i,t_1,\cdots,t_{i-1}) \cdot f_{n-i-1}(y_{i+1},\cdots,y_n,t_{i+1},\cdots,t_{n-1})$$

$$\text{if} \quad t_i = 1 .$$

b) $Sf : SY \longrightarrow SZ$ extends to $BY \longrightarrow BZ$.

c) There exists a commutative diagram

$$\begin{array}{ccc} WY & \searrow^{h} & \\ \downarrow & & \\ Y & \xrightarrow{f} & Z \end{array}$$

where $WY \longrightarrow Y$ is the standard retraction [Floyd] and h is a homomorphism.

d) f can be factored up to homotopy as $Y \longrightarrow Y_1 \longleftarrow Y_2 \longrightarrow \cdots \longrightarrow Z$ where the Y_i are also monoids and the maps are homomorphisms and the maps $Y_{2i} \longrightarrow Y_{2i-1}$ are homotopy equivalences.

In particular we can ask if $\text{ad}\,\tau : \Omega X \longrightarrow F^F$ is shm. Repeated use of the CHP provides the adjoint maps

4) $$\tau_n : F \times (\Omega X)^n \times I^{n-1} \longrightarrow F$$

as desired. Details are given in [10]. The significance of these maps is that they completely determine the fibration as we now indicate.

Let us back up a little. If a group G acts on a space Y, we can look at the orbit space Y/G. If $G \longrightarrow Y \longrightarrow Y/G$ is not a principal G-bundle, we can replace it, up to homotopy, by one, namely $G \longrightarrow EG \times Y \longrightarrow EG \times_G Y = Y_G$ where EG is the universal (contractible) G-bundle.

For any fibre space $F \longrightarrow E \longrightarrow X$, we have the fibration (up to homotopy) $\Omega X \longrightarrow F \longrightarrow E$ which suggests trying to identify E as $F_{\Omega X}$

in some sense. The lack of transitivity is a problem, so let us look at Y_G in more detail. One way of describing Y_G is a realization of the simplicial space

$$\overset{\to}{\underset{\to}{\overset{\to}{\to}}} Y \times G \times G \;\overset{\to}{\underset{\to}{\to}}\; Y \times G \;\underset{\text{proj}}{\overset{\text{action}}{\rightrightarrows}}\; Y .$$

In May's notation, the realization is $B(Y,G,*)$, though we have not mentioned degeneracies and prefer to avoid their use, cf. [7].

Now suppose that we have a sh-action of a monoid G on Y (i.e., maps $m_n : Y \times G^n \times I^{n-1} \longrightarrow Y$ adjoint to an shm-map). Form $\coprod_{n \geq 0} Y \times G^n \times I^n$ and factor by the following equivalence relation:

$$(y,g_1,\cdots,g_n,t_1,\cdots,t_n) \sim (\cdots,g_i g_{i+1},\cdots,\hat{t}_i,\cdots) \quad \text{if} \quad t_i = 0$$

$$\sim (m_i(y,\cdots,g_i,t_1,\cdots,t_{i-1})g_{i+1},\cdots,g_n,t_{i+1},\cdots) \quad \text{if} \quad t_i = 1$$

Again call the result Y_G or $B(Y,G,*)$.

In particular all this applies to a transport τ for $F \longrightarrow E \longrightarrow B$.

<u>Theorem</u>: Let $\{\tau_i\}$ be a family of maps satisfying 1), 2), 3) and 4). The map $B(F, \Omega X,*) \longrightarrow B(*, \Omega X,*) = B\Omega X$ is a quasifibration with fibre F. (With extra connective tissue, Fuchs has been able to

build an equivalent Dold fibration [3].)

If τ_i is obtained from $F \longrightarrow E \longrightarrow B$ using the CHP as indicated above, then $E \longrightarrow B$ is weakly fibre homotopy equivalent to $B(F, \Omega X, *) \longrightarrow B(*, \Omega X, *)$.

If $\{\tau_i\}$ is arbitrary as above and $\{\tau_i'\}$ is constructed from $B(F, \Omega X, *) \longrightarrow B(*, \Omega X, *)$ using the CHP, then $\{\tau_i\}$ is homotopic to $\{\tau_i'\}$.

Thus $\{\tau_i\}$ is a complete invariant of $E \longrightarrow B$; homotopy classes of transports classify fibrations.

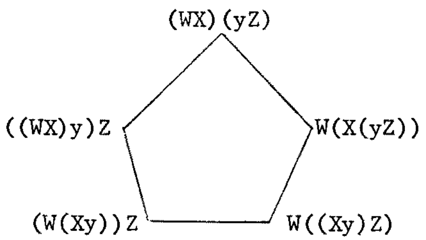
The usual way of classifying fibrations is by homotopy classes of maps $X \longrightarrow BH(F)$ where $H(F)$ is the monoid of self-homotopy equivalences of F. Now $\{ad\ \tau_i\}$ is an shm-map of ΩX into $H(F)$ and hence induces a map at the B level. We have thus

$$X \simeq B\Omega X \xrightarrow{Bad\tau} BH(F)$$

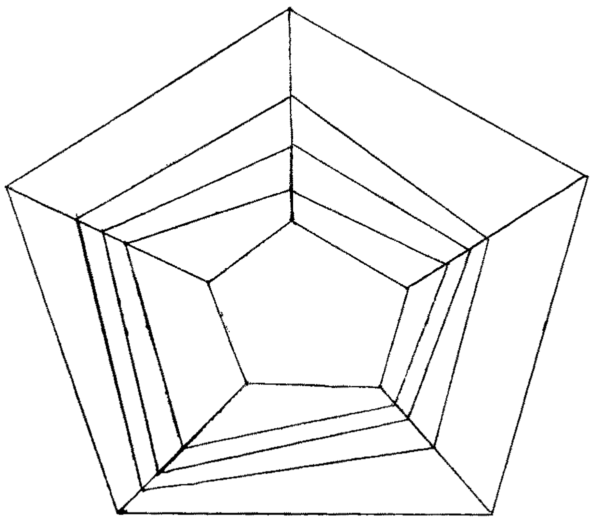
<u>Theorem</u>. For a suitable choice of the equivalence $X \simeq B\Omega X$, the classifying map above is the usual one [11].

Here we should note that we assume X has the homotopy type of a CW-complex in order to assert $X \simeq B\Omega X$. I am unaware of any study of more general topological conditions (e.g., perhaps weakly locally contractible and paracompact) which would guarantee the same equivalence.

Remarks on operads: Within the context of this conference, it is appropriate to mention the relation between the structures we have been studying and the concept of operads. Our transport $\{\tau_i\}$ is a collection of higher homotopies i.e., maps $F \times (\Omega X)^n \times I^{n-1} \to F$, whereas an operad action is of the form $Y^n \times M(n) \longrightarrow Y$, where $M(n)$ is a parameter space frequently more complicated than a cube, though often contractible in cases of current interest. An "ancient" example are my complexes K_n e.g., $K_3 = I$ but K_4 is a pentagon



and K_5 a polyhedron with 6 pentagonal and 3 quadrilateral faces. Malraison has a function space equivalent of K_n, readily described in terms of maps $[0,1] \xrightarrow{f} [0,n]$. Thinking of $f^{-1}(i)$ as dividing $[0,1]$ into n pieces, we can see the relevance to loop spaces by using loops parameterized from 0 to 1 and the classical addition of loops. The corresponding K_n structures can be pictured



One reason for studying $\{K_n\}$-spaces rather than strict monoids is that the definition is homotopy invariant. If $X \simeq Y$ and X is a monoid, Y need not admit an equivalent monoid structure (cf. Exotic multiplications on S^3 [Slifker]) but Y will admit an equivalent $\{K_n\}$-structure (usually called strongly homotopy associative - s.h.a.).

Now recall that an operad is, among other things, a category; where defined, composition is associative. It makes sense to talk of $M \longrightarrow \mathrm{End}\, X$ being shm rather than a strict morphism, a sh-functor rather than a strict functor. Again if $X \simeq Y$ and X is an M-space, then Y is at least an sh-M-space (Lada). Alternatively Boardman asserts Y is a WM-space where WM is his construction, presented here by Floyd.

Floyd has also pointed out that a WM - space X can be replaced up to homotopy by an M - space. Lada has given an alternative description of this process, namely $B(M,M,X)$ where B is constructed using cubes as above. Actually Lada, following May, usedthe associated triple MX which is just the free gadget

$$MX = \coprod M(n) \times_{\Sigma_n} X^n/\sim$$

where the equivalence is given entirely in terms of degeneracies $d_i : M(n) \longrightarrow M(n-1)$ corresponding to $X^{n-1} \longrightarrow X^n$ by inserting the base point in the i-th coordinate.

In comparing operads by morphisms $M \longrightarrow M^1$ which are homotopy equivalences on each component, we find the inverse maps $M^1 \longrightarrow M$ are at least shm. Finally since operads have associative compositions, we can generalize to sh - operads having operads act on operads.

Since the conference, I have seen work of Segal in which he has related E_∞ - Σ - operads to his Γ - structures and given an alternate approach to the last two paragraphs using essentially form d above for handling sh - morphisms.

To come back to more concrete objects, I will consider briefly the "local" approach to classification. Here local refers to structure defined on a space in terms of an open cover $\{U_\alpha\}$. For example, a fibre bundle $p:E \longrightarrow B$ is defined in terms of local product structures:

$$\begin{array}{ccc} p^{-1}(U_\alpha) & \equiv & U_\alpha \times F \\ & \searrow \quad \swarrow & \\ & U_\alpha & \end{array}$$

A fibre space over a nice base [1] can be defined in terms of local equivalences:

$$\begin{array}{ccc} p^{-1}(U_\alpha) & \simeq & U_\alpha \times F \\ & \searrow \quad \swarrow & \\ & U_\alpha & \end{array}$$

A foliation is defined in terms of special local coordinates:

$$U_\alpha \equiv R^k \times R^{n-k}$$

Now an open cover $\{U_\alpha\}$ gives rise to a simplicial space U:

$$\begin{smallmatrix}\to\\\to\\\to\\\to\end{smallmatrix} \{U_\alpha \cap U_\beta \cap U_\gamma\}_{\alpha,\beta,\gamma} \begin{smallmatrix}\to\\\to\\\to\end{smallmatrix} \{U_\alpha \cap U_\beta\}_{\alpha,\beta} \begin{smallmatrix}\to\\\to\end{smallmatrix} \{U_\alpha\}$$

where all intersections are non-empty. (If desired, think of $\{U_\alpha\}$ as a category U with $\mathrm{Ob}U = \coprod U_\alpha$, the disjoint union and $\mathrm{Mor}\,U$ given by $\mathrm{Mor}\,(x \in U_\alpha, y \in U_\beta) = \emptyset$ unless $x = y$ in which case $\mathrm{Mor}\,(x,x) = x$). There is an obvious map $BU = |U| \longrightarrow X$ which if X is paracompact is a homotopy equivalence. (The pictures in [9] are